

Corporate Finance in General Equilibrium models with Incomplete Markets: A Survey.¹

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Abstract

This is a survey of models of general equilibrium with incomplete markets and corporations. The intuition behind several of the difficulties that arise in these models is discussed. Interesting issues are pointed out also.

1 Introduction

In this survey we discuss several problems that arise when we consider a GEI (*General Equilibrium with Incomplete Markets*) economy with production, in which each firm is a corporation. A precise definition of a *corporation* is given in [19, Chapter 6] and this is the definition that we shall adopt. A corporation may be defined as a firm with three properties:

1. its capital (ownership) is divided into shares (equity contracts);
2. each shareholder is liable for the company's debts up to an amount limited to the current value of his equity share;
3. the company's equity contracts are traded on a stock market.

The third property is essential to the definition of corporation since it is what distinguishes a corporation from other structures of ownership like sole proprietorships and partnerships. The possession of a positive amount of a firm's share (equity contract), represents a commitment from the owner to the corporation. However, this commitment is short term in nature, in the sense that every owner of the firm's shares can decide to sell those shares at any given date.

The structure of ownership of the corporation brings the phenomenon of separation of ownership and control. Now the firm is owned by a large number of shareholders which are changing at any given date, so it would be impractical to have *all* of them run the firm on a daily basis. It is however supposed that managers of the firm, run it in the best interest of shareholders. This will be given a precise meaning when we discuss the objective function of the firm.

Even if managers act in the best interest of shareholders, it remains to define clearly *which* shareholders. Decisions of production and investment may carry long time consequences and the set of shareholders at the moment those decisions are made, may differ substantially to the set of shareholders at the moment in which those plans are implemented.

In this survey, we present examples of basic GEI models with firms. We use these models to exemplify and discuss the intuition of the basic problems that arise when dealing with this kind of models. The papers discussed here, constitute the *classical* literature in these subjects. The second section of this paper introduces the models that we use as examples and describe the problem of the firm, that is, which should be the firm's objective and how to achieve it. The third section presents some of the properties of GEI models with firms and discusses the classical literature on these subjects. Section four concludes.

2 The problem of the firm

2.1 Two GEI models with firms

While trying to give a panoramic view of the literature in the subjects that concern us, we will pay special attention to two models: a basic model presented in [19, Chapter 6], and a model extracted from [2]. This will allow us to get some insight into the conceptual and technical difficulties that are to be discussed later. It will also help us to provide concrete examples.

2.1.1 Basic model

This is a two-period economy where dates are numbered 0 and 1. At date 1, there are $S \geq 1$ possible states of nature whose purpose is to account for future uncertainty. For convenience we will refer to date $t = 0$ as state $s = 0$. There is a finite set of consumers indexed by $I = \{1, \dots, I\}$ and a finite set of firms indexed by $K = \{1, \dots, K\}$. There is a single consumption good at each state of nature, and so, a consumption plan for consumer i , denoted by x^i , is a vector in $\mathbb{R}_+^{S+1} = \{(x_1, \dots, x_{S+1}) \in \mathbb{R}^{S+1} | x_s \geq 0, s = 1, \dots, S+1\}$.

Consumer i is characterized by a utility function $u^i : \mathbb{R}_+^{S+1} \rightarrow \mathbb{R}$ and initial endowment $\omega^i \in \mathbb{R}_+^{S+1}$, for $i \in I$. Firms are described by K exogenously given technology sets: $Y^k \subset \mathbb{R}^{S+1}$ describes the income streams $y^k = (y_0^k, y_1^k, \dots, y_S^k)$ generated by investment projects available to “firm” k , for $k \in K$. Note that from the beginning, we assume the existence of these firms and are not concerned in how they came to be. Later, when we endow the firm with a more definite structure of ownership (a corporation), our only interest will be in the *operation* of these firms, and not in their possible creation or destruction. To ensure the existence of an equilibrium, we will make the following assumptions on the technology sets:

Assumption T. The technology sets Y^k ($k \in K$) have the following properties¹:

1. $Y^k \subset \mathbb{R}^{S+1}$ is closed
2. Y^k is convex
3. $Y^k \supset \mathbb{R}_-^{S+1}$
4. $Y^k \cap \mathbb{R}_+^{S+1} = \{0\}$
5. $(\omega + \sum_{k=1}^K Y^k) \cap \mathbb{R}_+^{S+1}$ is compact for all $\omega \in \mathbb{R}_+^{S+1}$

The financial structure of the economy, as a whole, is represented by a bond market on which J bonds are traded and a stock market on which the equity contracts for the K corporations are traded.

In the bond market, R_s^j denotes the exogenously given payoff of the j^{th} bond in state $s \in S \setminus \{0\}$ and $R_s = (R_s^1, \dots, R_s^J)$, $s = 1, \dots, S$ denotes the vector of payoffs of the J

¹These assumptions are standard in general equilibrium theory (see, for example, [4, Chapter 3]).

bonds in state s . Let $p = (p_1, \dots, p_J)$ represent the vector of prices for the J bonds. A portfolio of the J bonds is represented by $b = (b_1, \dots, b_J)$.

In the equity market, let $\theta_0^i \in \mathbb{R}_+^K$ represent the vector of initial ownership shares of agent i in the K firms. The ownership shares $\theta_0 = (\theta_0^1, \dots, \theta_0^I)$ are assumed to satisfy $\sum_i \theta_0^i = 1 \in \mathbb{R}^K$. That is, the outstanding amount of equity is normalized to 1. Transactions on the stock market are assumed to be costless, so we may assume that agent i sells his portfolio $\theta_0^i \in \mathbb{R}^K$ of initial ownership and purchases the new portfolio $\theta^i \in \mathbb{R}^K$. The vector of prices on the K equity contracts is denoted by $q = (q_1, \dots, q_K)$.

2.1.2 Stylized model

This model is very similar to our basic model and we maintain, whenever possible, the same notation. Again, we consider two dates and S different states of nature at date 1. Now, $I = \{1, \dots, I\}$ is a set of indexes for the *types* of consumers, so there are I different types of consumers. The same is valid for firms but now $K = 1$ so that there is only one type of firm. There is a continuum of firms, of unit mass, as well as a continuum of consumers of each type i , also set to have unit mass.

Consumers are again characterized by utility functions $\mathbb{E}u^i : \mathbb{R}_+^{S+1} \rightarrow \mathbb{R}$ and initial endowment $\omega^i \in \mathbb{R}_+^{S+1}$, for $i \in I$. Function u^i is assumed to be continuously differentiable, increasing and concave.

Instead of production sets² Y , firms are described by the function $f(k, \phi; s) : K \times \Phi \times \{1, \dots, S\} \rightarrow \mathbb{R}$, where $k \in K \subset \mathbb{R}_+$ is the amount of input invested in capital at date 0, ϕ is a technology choice affecting the stochastic structure of the output at date 1. We assume that f is continuously differentiable, increasing in k and concave in k, ϕ . Finally, Φ and K are closed, compact subsets of \mathbb{R}_+ and $0 \in K$.

Firms take both production and financial decisions, and their equity and debt (in bonds) are the only assets in the economy. Referring to our basic model, $J = 1$. Since the total amount of equity is, at the beginning, normalized to 1 ($\sum_i \theta_0^i = 1$), we will assume that this is kept constant so that the firm's capital structure is only given by the decision concerning the amount B of bonds issued, which will equate the firm's debt/equity ratio. The problem of the firm consists in the choice of its production plan k, ϕ and its financial structure B .

Again, the price of the bonds is represented by $p \in \mathbb{R}$ and the price of the equity by $q \in \mathbb{R}$.

2.2 The firm's objective function

Consider our basic model, we begin by supposing that the production plan is financed by the original shareholders, so if agent i is an initial shareholder of firm k , he contributes $y_0^k \theta_{0,k}^i$ as his share of the input costs of the firm. The budget constraints of agent i at dates 0 and 1, respectively, are given by:

²We dispense with the index since there is only one type of firm, and use k instead to represent the amount of input invested in capital at time 0.

$$x_0^i - \omega_0^i = -pb^i + (y_0 + q)\theta_0^i - q\theta^i \quad (1)$$

$$x_s^i - \omega_s^i = R_sb^i + y_s\theta^i, s = 1, \dots, S \quad (2)$$

The term $-pb^i$ represents the cost incurred by the agent at date 0 to buy portfolio b^i of bonds, term $q\theta_0^i$ corresponds to the income obtained by the sale of its initial portfolio of shares and $q\theta^i$ the cost of buying new portfolio θ^i . At date 1, state s of nature, Agent i receives the income R_sb^i from his bonds portfolio, and income $y_s\theta^i$, which corresponds to his share of the production of all firms from which he purchased equity shares at date 0.

For now, we bypass the problem of the separation of ownership and control by assuming that owners (original shareholders) of firm k , are directly involved in the process of deciding the production plan that the firm should adopt. Each shareholder will consider two ways in which the adopted production plan will affect his income: first, at date 0 he will get the income $(y_0 + q)\theta_0^i$, which also represents the (date 0) value of the firm, second, he also has rights to income stream across the $S + 1$ states $[-q_k, \hat{y}^k]'\theta_k^i$, where $\hat{y}^k = (y_1^k, \dots, y_S^k)$ represents the income stream generated by production in the different states of nature at date 1.

The first term represents the market value of the firm at date 0, the second term represents the *insurance* services provided by the portfolio of equities θ^i at date 1, that is, once production plans are fixed, firms equity shares have the same function of transferring wealth among the different possible states of nature that any other financial asset has. These possibilities of insurance are called the *spanning services* of the firm's equity contract.

Most of literature has as an objective function the maximization of the market value of the firm. This means, ignoring the second way in which the production plan affects income (the spanning services). Different justifications for this can be found in the literature.

One of these justifications if formalized in what is called the *spanning property* or the *partial spanning assumption*. Capital Asset Pricing Model has this property, it has it in the space of mean and variance, which in that model are the determinants of utility. For a discussion on the subject a good source is Drèze [9]. Other examples of this justification are found in Diamond [7] and in Ekern [11]. The intuition behind the reason for which this restriction (the spanning condition) leads us to adopt market value maximization, is that changes in the equity contract of a particular firm will not affect the insurance possibilities (the spanning services) offered in the stock market. A firm cannot create new spanning opportunities by altering its production plan, so .

A second justification found in the literature is the one provided in Grossman and Hart [14]. The main assumption, known as *competitive price perception* is that each consumer considers the benefit obtained from purchasing an equity share θ_k^i , exactly compensated by its price q_k , then, clearly, the spanning term will not affect the agent's choice of production plan.

If either of these assumptions hold, maximization of the market value of the firm is justified as the objective each agent would pursue when considering the production plan to be implemented. This market value, given by the expression $(y_0 + q)$, depends on the price of the firms' equities. But, what is or should be the price of the equity contract of the firm? This question is intimately related to the objective of the firm. After all, agents will be able to maximize the value of the firm only if they know how to calculate this value. This relationship is discussed further in the following subsection.

2.3 Pricing functions

When considering the market value of the firm, and the decision of which production plan to adopt, agents (initial shareholders) must take into account the price of the firm's equity to determine this value, however, depending on which production plan they choose, the equity contract of the firm, traded in the stock market, becomes a different financial asset, since it generates a different stream of income across the states of nature at date 1, and so, it provides different insurance opportunities.

What agents need is a way to determine, or guess, which should be the price of the firms equity depending on the production plan adopted. This can be done without ambiguity when markets are complete and this case will be discussed briefly. We are most interested, however, in the case of incomplete markets since, after all, the assumption of incomplete markets seems much more reasonable.

2.3.1 Complete markets

If markets are complete, the price of any cash flow generated by a production plan, can be determined unambiguously. Intuitively, given any income stream at date 1, a portfolio using other assets (bonds) of the financial market can be constructed, such that it pays the same income stream at date 1. The prices of those other assets (bonds) are known and the price of the portfolio is well determined. Since the original income stream and the portfolio provide the same insurance possibilities, their prices should be equal if there are no arbitrage possibilities in the economy (a necessary condition for the existence of competitive equilibrium).

Being a bit more formal, the absence of arbitrage implies the existence of S state prices³. These state prices, $\pi = (\pi_1, \dots, \pi_S)$ can be used to evaluate the price of any income stream $\tilde{y} \in \mathbb{R}^S$ at date 1 in the following manner:

$$q_k = \sum_{s=1}^S \pi_s \tilde{y}_s \quad (3)$$

Moreover, when financial markets are complete, the vector of state prices is uniquely determined, so evaluation of income streams (production plans) by all the initial shareholders of the firm (and all agents in the economy) will be the same. Maximization of the value of the firm, $(y_0^k + q)$, can be performed. It is easy to see that in this case, state

³See, for example, Cass [3].

prices are unique. Consider the matrix of payments of all (bonds and equities) assets in the economy:

$$[R \ y] = \begin{bmatrix} R_1^1 & \cdots & R_1^J & y_1^1 & \cdots & y_1^K \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R_S^1 & \cdots & R_S^J & y_S^1 & \cdots & y_S^K \end{bmatrix} \quad (4)$$

If financial markets are complete, $\text{rank}[R \ y] = S$. Since there is no arbitrage, using (3), $\pi[R \ y] = [p \ q]$, and there is a unique vector of state prices π given a vector of (observed) security prices $[p \ q]$.

2.3.2 Incomplete Markets

When markets are incomplete ($\text{rank}[R \ y] < S$), if there is no arbitrage, a vector of state prices still exists, but it is not uniquely determined. Intuitively, prices of existing assets might not be enough to determine without ambiguity the value of any income stream. An exception occurs when the *partial spanning* assumption is satisfied. We present this assumption as in Magill-Quinzii [19]:

Partial Spanning Assumption. There is a linear subspace $Z \subset \mathbb{R}^{S+1}$ whose dimension is at most K , such that $Y^k \subset Z$, $k = 1, \dots, K$.

If this assumption holds, it is clear that *any* production plan of firm k generates an income stream that can be represented or constructed as a portfolio of the other assets in the economy, the price is then unambiguously determined. In terms of state prices, even though different shareholders of firm k 's equity, might use different state price vectors to evaluate production plans (income streams), the resulting evaluation will be the same, regardless of the state price vector chosen. Again, maximization of the market value of the firm $y_0 + q$ can be performed, and it is clear to every agent which production plan is optimal in this sense, that is, the decision is unanimous. Unanimity is another subject that interests us and will be discussed latter.

In the literature, the spanning condition is introduced in [11] by Ekern and Wilson to identify conditions for unanimity. Radner, in [29], reinterprets Ekern and Wilson's analysis in "standard Arrow-Debreu" terms and provides conditions under which ex ante and ex post stockholders are unanimous in their decisions.

Now, consider the situation in which the spanning condition does not hold. In this case, the standard assumption made in the literature is that firms⁴ "guess" or conjecture a price function $q_k : \mathbb{R}^{S+1} \rightarrow \mathbb{R}$ for the price of firm k 's equity, depending on the production plan adopted $y^k \in \mathbb{R}^{S+1}$. How should this conjecture be made? Throughout the literature, we can find different answers.

Grossman and Hart introduce in [14] the following price function, expressed in the language of our basic model:

⁴Here, by *firms* we are referring to firms' management which is composed by shareholders or to act in shareholders' best interest.

$$q_k(y^k) = \sum_{i \in I_k} \theta_{0,k}^i \pi^i(\bar{x}^i) y^k, \forall y^k \in Y^k \quad (5)$$

where $I_k = \{i \in I | \theta_{0,k}^i > 0\}$ is the set of shareholders of firm k , $\pi^i(\bar{x}^i) = (\pi_1^i(\bar{x}^i), \dots, \pi_S^i(\bar{x}^i))$, and $\pi_s^i(\bar{x}^i) \equiv (\frac{\partial u^i(\bar{x}^i)}{\partial x_s^i}) / (\frac{\partial u^i(\bar{x}^i)}{\partial x_0^i}) = \overline{\text{MRS}}^i(s)$ is the marginal rate of substitution between present consumption and consumption at date 1, state s , and represents the own personal evaluation of state s by agent i . If each agent which is a stockholder of firm k , evaluates any future income stream y^k as⁵ $\sum_{s=1}^S \pi_s^i(\bar{x}^i) y_s^k$, equation (5) tells us that the firm evaluates the income stream y^k using a weighted average of the private evaluations of its (initial) stockholders, and using as weights the number of shares each stockholder possesses at the moment of the decision.

In the language of our stylized model, Grossman and Hart's pricing function could be written as

$$q_k(k, \phi, B) = \mathbb{E} \sum_i \theta_0^i \overline{\text{MRS}}^i(s) [f(k, \phi; s) - B], \forall k, B \quad (6)$$

A similar pricing function, introduced explicitly by Drèze [8] and previously implied by Diamond [7] is:

$$q_k(y^k) = \sum_{i \in I_k} \theta_k^i \pi^i(\bar{x}^i) y^k, \forall y^k \in Y^k \quad (7)$$

Where the only difference with Grossman-Hart's pricing function is that now it is the quantities of shares owned by *final* stockholders the ones used as weights for the private evaluations of future income streams. In the language of our stylized model:

$$q_k(k, \phi, B) = \mathbb{E} \sum_i \theta^i \overline{\text{MRS}}^i(s) [f(k, \phi; s) - B], \forall k, B \quad (8)$$

Of these latter pricing functions, which one is more appropriate? Grossman and Hart point out in their paper [14] that *conceptually*, the answer depends on how legally binding the decisions of initial shareholders are. They say that "In general, as ownership changes, there will be pressure on the firm to change its production plan. The incompleteness of markets at the initial date prevents agreement between all present and future owners as to the worth of contemplated production plans. As a result, the balance of power between shareholders at different dates in the determination of production decisions becomes crucially important".

Under the assumption that commitment to a production plan by initial shareholders is legally binding, Grossman and Hart's model, with its assumption of *competitive price perceptions* and its respective pricing function (which is a direct consequence of this assumption) has the advantage that it allows us to consider shareholders that take into account the effect of their decisions on the firm's equity price. This characteristic is missing in Diamond's and Drèze's model.

⁵This is the essence of the competitive price perceptions assumption made in [14].

A third kind of price function proposed in the literature corresponds to the one presented in Bisin et al. [2] from where we borrow our stylized model. Similar pricing functions appear in Makowski [20] and [21], in Makowski and Ostroy [22], in Allen and Gale [1] and Pesendorfer [28].

The seminal paper is Makowski [21] where he based his analysis on Joseph Ostroy's no *surplus characterization* of perfect competition⁶. The problem of timing and coordination between initial and final shareholder's does not arise since, in a sense, firms "choose" simultaneously shareholders and productions plans. This model, with its pricing function, has the desirable characteristic of unanimity among shareholders. This characteristic is lacking in Grossman and Hart's model, in which, shareholders agree in defining as objective of the firm the maximization of (current) market value, but disagree in which is the plan that maximizes this value. To solve this problem, Grossman and Hart introduce a scheme of side payments that facilitate an agreement.

In Makowski's model, initial shareholders sell their shares to individuals having the highest evaluation for the firm's income stream and ex-post (final) shareholders in each period form an homogeneous group. Makowski's pricing function is such that the equilibrium prices of shares in the firms are sufficiently high, so that all individuals except final shareholders do not want to buy firm's shares. In our stylized model:

$$q_k(k, \phi, B) = \max_i \mathbb{E}[\overline{\text{MRS}}^i(s)(f(k, \phi; s) - B)], \forall k, B \quad (9)$$

2.3.3 Equilibrium

Although we have discussed the basic setting of our two models, we still have not presented the appropriate definition of equilibrium. As it turns out, the "correct" definition of equilibrium depends on the assumptions we make and the pricing functions that we use. The basic definition of equilibrium is still behind each of these slightly more sophisticated definitions that we will consider: *An equilibrium is an allocation of consumption for every agent, such that every agent maximizes his utility, given his budget constraint; an allocation of production plans, such that each firm maximizes its current value; and a vector of prices for goods, bonds and equity shares such that goods, financial and stock markets clear.*

Consider first the cases of complete markets and of incomplete markets with partial spanning. In these cases, we can proceed in two steps. In the first step, we can consider an exchange equilibrium, that is, an equilibrium with fixed production plans $\bar{y} = (\bar{y}^1, \dots, \bar{y}^K)$. Indeed, if production plans are fixed, equity shares of the different firms play the same role as bonds. Grossman and Hart call this, the *security role* of the shares⁷. In the second step, we ask that each firm maximizes its current value. In the language of our basic model:

⁶See, for example, [26] and [27].

⁷As opposed to their *ownership role*, which simply consists on the claims of the shareholders on the production plans of the firms.

Definition 1. Let $\mathcal{E}(u, \omega, \theta_0, R, Y)$ denote an economy with complete markets or an economy with incomplete markets that satisfies the partial spanning assumption. A vector of allocations, portfolios, production plans and prices, $((\bar{x}, \bar{b}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}))$ (with $\text{rank}(\bar{y}) = \dim Z$ whenever markets are incomplete), is a *stock market equilibrium* if, given \bar{y} ,

- (i) $(\bar{x}^i, \bar{b}^i, \bar{\theta}^i) \in \arg \max \{u^i(x^i) | (x^i, b^i, \theta^i) \in \mathbb{B}(\bar{p}, \bar{q}, \tilde{\omega}^i, R, \bar{y})\}$, $i = 1, \dots, I$.
Where $\mathbb{B}(\bar{p}, \bar{q}, \tilde{\omega}^i, R, \bar{y}) = \{x^i \in \mathbb{R}_+^{S+1} | x^i - \tilde{\omega}^i = W(b, \theta - \theta_0)'\}$, $\tilde{\omega} = \omega + \sum_{k=1}^K \theta_{0,k} y^k$,
and

$$W = \begin{bmatrix} -p_1 & \cdots & -p_J & -q_1 & \cdots & -q_K \\ R_1^1 & \cdots & R_1^J & y_1^1 & \cdots & y_1^K \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R_S^1 & \cdots & R_S^J & y_S^1 & \cdots & y_S^K \end{bmatrix}$$

- (ii) $\sum_{i=1}^I (\bar{b}, \bar{\theta} - \theta_0)' = 0$, and, \bar{y} is such that
(iii) $\bar{y}^k \in \arg \max \{\pi y^k | y^k \in Y^k\}$, $k = 1, \dots, K$, where π is a no-arbitrage state price vector.

In this definition, (i) and (ii) correspond to an exchange equilibrium for fixed \bar{y} . Maximization of the value of the firm is required in (iii). Note that if markets are complete, π in (iii) is unique.

If markets are incomplete, and the competitive price perceptions assumption is satisfied, we proceed in a similar manner to the previous case. Again, in the language of our basic model:

Definition 2. Let $\mathcal{E}(u, \omega, \theta_0, R, Y)$ denote an economy with corporations in which the competitive price perceptions assumption holds. A vector of allocations, portfolios, production plans and prices, $((\bar{x}, \bar{b}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}))$, with $\text{rank}(\bar{y}) = \dim Z$, whenever markets are incomplete, is a *stock market equilibrium* if, given \bar{y} ,

- (i) $(\bar{x}^i, \bar{b}^i, \bar{\theta}^i) \in \arg \max \{u^i(x^i) | (x^i, b^i, \theta^i) \in \mathbb{B}(\bar{p}, \bar{q}, \omega^i, \theta_0^i, R, \bar{y})\}$, $i = 1, \dots, I$.
Where $\mathbb{B}(\bar{p}, \bar{q}, \omega^i, \theta_0^i, R, \bar{y}) = \{x^i \in \mathbb{R}_+^{S+1} | x^i - \omega^i = (y_0 + q)\theta_0^i e^0 + W(b, \theta - \theta_0)'\}$, and
 $e^0 = (1, 0, \dots, 0) \in \mathbb{R}_+^{S+1}$.

- (ii) $\sum_{i=1}^I (\bar{b}, \bar{\theta} - \theta_0)' = 0$, and, \bar{y} is such that
(iii) $\bar{y}^k \in \arg \max \{\sum_{i \in I_k} \theta_{0,k}^i \pi^i(\bar{x}^i) y^k | y^k \in Y^k\}$, $k = 1, \dots, K$.

Again, (i) and (ii) can be viewed as corresponding to an exchange equilibrium for fixed \bar{y} , and (iii) as the maximization of the value of the firm using Grossman and Hart's pricing function.

Finally, we will define the appropriate concept of equilibrium when considering Makowski's pricing function. One of such economies is the one of our stylized model:

Definition 3. A competitive equilibrium for the stylized economy is a collection $(\bar{k}, \bar{\phi}, \bar{B}, (\bar{c}^i, \bar{\theta}^i, \bar{b}^i)_{i, \bar{p}, \bar{q}, q(\cdot)})$, such that:

(i) $(\bar{k}, \bar{\phi}, \bar{B})$ solve the problem of the firm:

$$\begin{aligned} & \max_{k, \phi, B} -k + q(k, \phi, B) + pB \\ & \text{subject to} \\ & f(k, \phi; s) \geq B, \forall s \in S \end{aligned}$$

(ii) for all i , $(\bar{c}^i, \bar{\theta}^i, \bar{b}^i)$ solve consumer i 's problem:

$$\begin{aligned} & \max_{\theta^i, b^i, x^i} \mathbb{E}u^i(x^i) \\ & \text{subject to} \\ & x_0^i - \omega_0^i = [-\bar{k} + \bar{q} + \bar{p}\bar{B}]\theta_0^i - \bar{q}\theta^i - \bar{p}b^i \\ & x_s^i - \omega_s^i = [f(\bar{k}, \bar{\phi}; s) - \bar{B}]\theta^i + b^i, \forall s \in S \\ & b^i \geq 0, \theta^i \geq 0, \forall i \end{aligned}$$

(iii) markets clear:

$$\begin{aligned} \sum_i \bar{b}^i & \leq B \\ \sum_i \bar{\theta}^i & \leq 1 \end{aligned}$$

(iv) the equity price map satisfies:

$$\begin{aligned} q(\bar{k}, \bar{\phi}, \bar{B}) & = q, \\ q_k(k, \phi, B) & = \max_i \mathbb{E}[\overline{\text{MRS}}^i(s)(f(k, \phi; s) - B)], \forall k, B \end{aligned}$$

Finally, we note (and do not prove) that for both of the economies considered, and for all the definitions of equilibrium presented, a competitive equilibrium exists.

3 Properties of the models

Now that we have addressed some of the first difficulties encountered when dealing with GEI models with firms, we pass to describe some of the most important and interesting features of these models.

3.1 Unanimity

We have discussed which should be the appropriate objective of the firm and presented several pricing functions. To support the use of any of these building blocks in a model, a coherent story must be told about the decision process through which the shareholders of a firm reach an agreement on how to carry out the firm's objective. One of such

stories, that has the nice feature of being simple (although arguably unrealistic in many settings) is that the decision by the shareholders is unanimous.

As mentioned before, in Diamond [7] unanimity is obtained, but it was in Ekern and Wilson [11] that the relationship between unanimity and the spanning condition was explored. Later, Radner [29] studied Ekern and Wilson's model in a standard general equilibrium setting and stated the spanning condition in the form it was presented here.

In general, decisions are unanimous in models with partial spanning. The reason is simple and has been already mentioned: although different individuals might use different no-arbitrage state price vectors to evaluate date 1 income streams, evaluations are the same. Shareholders of a given firm form an heterogeneous group but nevertheless, agree on which production plan is the optimum.

The spanning condition is too restrictive. As Grossman and Hart [14] note: "If there are many more states of the world than firms, then it is very likely that some firm will be able to produce a vector of state contingent incomes which is not a linear combination of the existing vectors of state contingent incomes." In Grossman and Hart's model, decisions are not unanimous, instead, a different story is told: shareholders reach an agreement by recourse to a *side payments* scheme which in fact is a simplification of a more complex story of bids and takeovers. The moral is that in general, one should not expect unanimity.

Models that correspond to our third pricing function (Makowski's), have unanimity. The spanning condition is not satisfied in such models but, as noted before, given the pricing function of these models, only the agents who value the most the shares of the firm, are the ones that actually buy the shares at any given date. Unanimity comes from the fact that shareholders in these models conform an homogeneous group.

Hart [16] obtains an asymptotic result. He proves that for very large economies, "net market value maximization (approximately) represents the wishes of all initial shareholders". His result is not valid, however, for finite economies like the ones considered in our examples. Even in very large, but finite, economies, the result is not valid and we must seek comfort in the finding that in this case, "the gain to any one shareholder from a departure from net market value maximization is small".

3.2 Efficiency

Now we discuss the welfare properties of GEI models with firms. In general, equilibrium allocations in these models will not be optimal in the general Pareto sense, instead, most of the models mentioned so far are efficient in a more restricted sense known as *constrained Pareto optimality* and introduced in GEI models with firms by Diamond [7]. We present this concept in the context of our stylized model:

In our stylized model, a consumption allocation x^i is admissible if:

1. it is feasible, that is, there exists a production plan k, ϕ of firms such that:

$$\begin{aligned}\sum_i x_0^i + k &\leq \sum_i \omega_0^i \\ \sum_i x_s^i &\leq \sum_i \omega_s^i + f(k, \phi; s), \forall s \in S\end{aligned}$$

2. it is attainable with the existing asset structure, that is, there exists B and, for each consumer of type i , a pair θ^i, b^i such that:

$$x_s^i = \omega_s^i + [f(k, \phi; s) - B]\theta^i + b^i, \forall s \in S$$

Definition 4. A competitive equilibrium allocation is *constrained Pareto efficient* if we cannot find another admissible allocation which is Pareto improving, that is, that makes no agent worse-off and makes some agents (strictly) better-off.

Why should we expect less than Pareto efficiency? In [15], Hart provides counterexamples where equilibrium allocations were Pareto dominated by other equilibrium allocations⁸ The intuition is that consumers don't have available all insurance possibilities and some Pareto optimal allocations might be unreachable given the current financial structure. Later, Grossman [13] showed that equilibria in GEI models with firms are constrained Pareto efficient.

3.3 Modigliani-Miller

Modigliani and Miller [25] demonstrated the irrelevance of the financial structure of the corporation in a partial equilibrium model. Their result is valid in the absence of imperfections like taxes, transaction costs, bankruptcy costs, asymmetric information, etc. The result was extended later by Stiglitz [31] to a general equilibrium model with complete markets. Hellwig [17] examines again the validity of the result a GE model with bankruptcy. DeMarzo [5] showed that the result holds even if markets are incomplete, this was confirmed by Duffie and Shafer [10] and also mentioned by Magill and Quinzii in [19] from where we extract our basic model.

A version of the result also holds for Diamond's and Grossman and Hart's models. Regarding the stylized model in its original source [2], Modigliani-Miller theorem does not necessarily hold when there exist borrowing conditions.

A nice survey on the subject, that reviews the first 30 years after the Modigliani-Miller was first stated can be found in [24].

Gottardi [12] discusses the validity of the theorem when markets are incomplete, he concludes that "in presence of any type of derivative security a change in the capital structure of a firm will modify, generically, both the real equilibrium allocation and the value of the firm. The reason is that the payoff of the derivative securities is affected in a non-linear way by changes in the firm's financial policy; thus the set of the agents'

⁸Hart's counterexamples have multiple equilibria.

insurance opportunities is also modified". This result hampers the validity of Modigliani-Miller's theorem because, again in Gottardi's words, "...the irrelevance result originates from a fundamental linearity property of the problem, independent of the completeness of the market. The change in the return on equity, induced by a modification of the firm's capital structure, is in effect a linear combination of the returns on the assets traded by the firm. By the no arbitrage condition, the price of each asset is then also a linear function of the income it generates in the future states of nature. These two facts lie at the origin of the invariance both of the equilibrium allocation and of the firm's valuation". Gottardi's paper does not contradict DeMarzo's result because it considers derivatives in his model, a feature lacking in DeMarzo's.

An more recent extension to incomplete markets, infinite horizon economies, can be found in Thorsten [33]. More recent developments include [30].

The relevant conclusion is that if the Modigliani-Miller theorem holds in a General Equilibrium model, equilibria are indeterminate, since the financial structure of the firm is also indeterminate, however, departures from neo-classical assumptions usually have the consequence of making financial decisions of firms *relevant*, we have seen that this is the case with derivatives [12], bankruptcy [17], and borrowing constraints [2], it is also the case with taxes [23] and asymmetric information [18].

4 Conclusion

We have presented two examples of models of general equilibrium with incomplete markets and firms. The firms considered in these models and in most of the literature discussed, are corporations. This has implications for the (internal) organizational structure of the firms and for the markets in the economy (we assumed the existence of a stock market). A precise definition of corporation was given in the introduction.

The discussion was carried out at a basic level, trying to provide examples and intuition for the conceptual and technical difficulties that arise in these models, and how these difficulties have been addressed in the literature.

Some other issues with these models were left out of the discussion. For example, we did little mention of the literature of GEI models with asymmetry of information, default and collateral, which certainly brings new difficulties and interesting problems and for which there is a wealth of literature.

Also, we restricted our attention to models in which we assume competitive behavior. Other models have a more strategic approach or a mixture of general equilibrium for the economy and strategic behavior for the shareholders of the firm. See for example DeMarzo [6], which incorporates voting.

In general, one gets the sense that although most of the problems that arise in these models have been addressed, there is still no consensus among economists about which is the "best" way to model these economies. This is reasonable given the nature of the difficulties described. Take for example the different pricing functions we mentioned. Models with partial spanning have nice properties (unanimity, efficiency) but at a cost of imposing the condition that no firm can innovate and create new securities, an as-

sumption that seems implausible given the large number of states of nature and few number of firms one would expect to encounter in real economies. We can then turn to a Makowski-type pricing function, which also carries the nice property of unanimity but at the cost of having complete homogeneity in shareholders' marginal rates of intertemporal substitutions. If we allow for some heterogeneity among shareholders, like in Grossman and Hart's model, an equilibrium still exists but without unanimity and a (somewhat implausible) story of side payments, or of bids and takeovers, must be told to justify the result.

On the other hand, this lack of consensus and ongoing difficulties in GEI models with firms are good news for economists, they signal the existence of much work to do and interesting problems to study.

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